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THE MONOTONICITY OF THRESHOLD DETECTION PROBABILITY IN STOCHASTIC ACCUMULATION PROCESS

by

Joseph Kreimer[1]
and
Moshe Dror[2]

1 Industrial Engineering and Management, Ben-Gurion University of the Negev, Beersheva, Israel.

2 Centre de recherche sur les transports, Université de Montréal, Montréal, Canada, and INRS-Telecommunications, 3 Place du commerce, Verdun, Québec, Canada.

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ABSTRACT

The purpose of this paper is to examine the likelihood of exceeding a boundary value $A<\infty$ for a finite sum of independent, identically distributed random variables. More specifically, we calculate for a number of well-known distributions, the most likely sum to exceed A and examine the monotonicity of the probability to exceed A for a finite sum of such random variables.

In this paper we prove for a number of distributions that the probability for the value of the sum of the first k (but not before) of i.i.d.r.v. to exceed a given value A is monotonically increasing in the range $k<k^*$ (or $k<k^*+1$) where $k^*=\max k$ such that $k\mu<A$. We conjecture that this monotonicity property is preserved for a much larger family of distribution functions than those examined in the paper.

Key word : threshold probability.

RÉSUMÉ

Dans cet article, nous nous intéressons au comportement par rapport à une valeur cible A d'une somme finie de variables aléatoires indépendantes et identiquement distribuées avec moyenne μ.

Nous démontrons pour un ensemble de distributions usuelles que la probabilité que la valeur de la somme dépasse la cible pour la première fois après k termes est monotone croissante sur l'intervalle $k<k^*$ (ou $k<k^*+1$) où k^* est le plus grand entier k tel que $k\mu<A$. Nous faisons la conjecture que cette propriété de monotonie est satisfaite pour une famille de distributions qui dépasse largement celle considérée dans cet article.

Mot-clé : somme de variables aléatoires, valeur cible.

1. Introduction

1.1 General Problem Statement and Motivation

Given a sequence of independent, identically distributed, random variables $r_i; i = 1, 2, 3, \ldots$ with positive mean value μ and variance σ^2, together with a positive fixed value A (assume $A > \mu$). For this sequence of random variables $r_i; i = 1, 2, 3, \ldots$ denote by $P_A(k)$ the probability that the value of the sum of the first k (but not before) r_i's exceeds the value A.

$$P_A(k) = Prob.(\sum_{i=1}^{k-1} r_i \leq A < \sum_{i=1}^{k} r_i) \qquad (1.1)$$

We call the sequence $P_A(k); k = 1, 2, 3, \ldots$ the **threshold detection probabilities** for the given value A.

The threshold detection probability - (TDP), $P_A(k)$ provides a measure of likelihood of overstepping a boundary value A at exactly k^{th} trial when in successive steps accumulating the effects of a randomly sampled phenomena. Given a distribution of the values associated with the observed phenomena, one is interested in the characteristics of the TDP for a given value of A. More specifically, we examine the following questions:

(i) What is the most likely number of trials required to overstep the threshold value A ?

(ii) In what range of k is the sequence $P_A(k); k = 1, 2, 3, \ldots$ monotonically increasing (decreasing) ? (This second question can be also stated in terms of convexity (concavity) of the sum $\sum_k P_A(k)$.)

We restrict the analysis to a number of very familiar and relatively simple distributions.

For the sequence of i.i.d.r.v. $r_i; i = 1, 2, 3, \ldots$, denote by $k^* = \max k$ such that $k\mu \leq A$. (obviously $(k^* + 1)\mu > A$).

In this paper we prove for a number of the distributions examined, that the sequence $P_A(k); k = 1, 2, 3, \ldots$ is monotonically increasing in the range $k \leq k^*$ (or $k \leq k^* + 1$). For some of the distributions we also prove that the most likely number of trials required to overstep the threshold value A is k^* (or $k^* + 1$). In other words $P_A(k^*) = \max_k P_A(k)$ or $P_A(k^* + 1) = \max_k P(k)$.

This research was motivated by the realization that answers to questions (i) and (ii) have some important implications (and applications) in a number of unrelated real-life scenarios: Signal detection in Radar Systems, Signal Processing in Physiology and Medicine, Reliability and Inventory Systems, and more. The questions (i), (ii) were examined for the following distributions: Bernoulli trials, Exponential, Normal, and Poisson. For the above distributions , the implications of determining the most likely number of trials to overstep a given threshold value is demonstrated for at least one real life case. We close this section with the conjecture that the monotonicity property of the sequence $P_A(k); k = 1, 2, 3, \ldots$ is preserved under very mild

assumptions on the underlying cumulative distribution function from which the r_i's are drawn.

Conjecture 1 : Given *any* cumulative distribution function with mean $\mu > 0$, variance σ^2, and a fixed value $A > \mu$. In addition, assume that the coefficient of variation $\sigma/\mu \leq 1$ and that the mean μ is greater or equal to the median. (For some distributions these restrictions are not necessary.) Then the sequence $P_A(k); k = 1, 2, 3, \ldots$ is monotonically increasing in the range $k \leq k^*$.

1.2 Notations

Denote by [a] the largest integer less than or equal to the real number a.

By $D_A(k)$ denote the following difference:

$$D_A(k) = P_A(k+1) - P_A(k) \qquad (1.2)$$

Since the distribution of the sum of i.i.d.r.v. is given by convolution, (1.1) can be rewritten as:

$$P_A(k) = F^{(k-1)}(A) - F^{(k)}(A) \qquad (1.3)$$

where $F^{(i)}(A)$ denotes the i-times convolution of the cumulative distribution function $F(x)$.

We restrict our subsequent discussion to TDP defined by (1.3).

Any additional notational conventions will be defined as needed.

2. Bernoulli Trials

2.1 Mathematical Background

The first probability distribution we examine is that of Bernoulli Trials.

Suppose that a trial whose outcome can be classified as either a 'success' or as a 'failure' is performed. Let r_i denote the outcome of the i^{th} trial and take the value 1 if it is a success and value 0 if it is a failure. In this case there is no sense to consider noninteger A, thus $A = [A]$.

$r_i = 1$ with probability p and $r_i = 0$ with probability $q = 1 - p$. The mean value of this experiment $\mu = p$ and $P_A(k)$ is defined as follows:

$P_A(k) = 0$ if $k \leq A$

$$P_A(k) = \binom{(k-1)}{A} p^A q^{k-1-A} p = \binom{(k-1)}{A} p^{A+1} q^{k-1-A}, \quad \text{if } k > A$$

In this experiment we count the number of successful trials and examine their probability of crossing a certain given value A exactly at the k^{th} trial.

Theorem 1 : The most likely number of Bernoulli trials required to cross the given threshold value A is $k^* + 1$. $P_A(k)$ is nondecreasing for $k \leq k^* + 1$ and nonincreasing for $k > k^* + 1$.

Proof:

Consider the difference $D_A(k)$. If $k + 1 \leq A$, then $P_A(k) = 0$, and obviously $D_A(k) \geq 0$. Thus, $P_A(k)$ is nondecreasing in k. If $k > A$, then:

$$D_A(k) = \binom{k}{A} p^A q^{k-A} - \binom{(k-1)}{A} p^A q^{k-1-A} =$$

$$= \frac{k!}{A!(k-A)!} p^A q^{k-1-A} \left(\frac{kq}{k-A} - 1 \right) =$$

$$= \frac{(k-1)!}{A!(k-1-A)!} p^A q^{k-1-A} \left(\frac{A-kp}{k-A} \right)$$

Therefore $D_A(k) \geq 0$ if and only if $kp \leq A$. Thus, $P_A(k)$ is nondecreasing for $k \leq k^* + 1$ and nonincreasing for $k > k^* + 1$, which results in $k^* + 1$ being the point of maximum for $P_A(k)$ •

2.2 Applications

Example 1 : Signal Detection and Verification in Radar Systems. The Most Likely Number of Periods till Signal Verification.

An important stage of signal processing in radar systems is signal detection and verification. (See for example Skolnik, 1962 for more details.)

These systems periodically search for detection according to a certain frequency. Detections of a specific target appear in each period according to Bernoulli trials with probability p of success. This probability is generally a function of signal-to-noise ratio (denoted by S/N) and can be perceived as constant during certain time intervals.

A target is declared to be verified if during a certain given number of periods (M) the number of detections was greater than some given integer A - verification threshold.

An operator of such systems is interested in the most likely number of periods till the verification of signal detection. It is readily seen that this problem is described exactly in terms of the formulas (1.1) and (2.1) and therefore the most likely number of periods till target verification is given by $k^* + 1$.

3. Exponential Distribution

3.1 Mathematical Background

Assume that a sequence of random variables $r_i; i = 1, 2, 3, \ldots$ are independently drawn from an exponential cdf $F(x)$ with parameter $\lambda, \lambda \geq 0$.

$$F(x) = 1 - e^{-\lambda x} \text{ for } x \geq 0$$

$$F(x) = 0 \text{ for } x < 0 \qquad (3.1)$$

The expected value of each of these random variables r_i is $\mu = \frac{1}{\lambda}$.

Theorem 2 : Given that the i.i.d.r.v. r_i are exponentially distributed according to (3.1), then the probability $P_A(k)$ defined by (1.1) attains its maximal value at $k = k^* + 1$ where k^* is given by $k^* \frac{1}{\lambda} \leq A < (k^* + 1)\frac{1}{\lambda}$. Moreover, $P_A(k)$ is nondecreasing for $k \leq k^* + 1$ and nonincreasing for $k > k^* + 1$.

Proof : Observe that a sum of n i.i.d.r.v. drawn from an exponential cdf $F(x)$ is distributed according to an n-Erlang distribution. I.e.,

$$f_{r_1+r_2+\ldots+r_n} = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, \quad x \geq 0 \qquad (3.2)$$

The corresponding cdf is denoted in the literature as:

$$Er_n(x) = 1 - \sum_{i=1}^{n-1} e^{-\lambda x} \frac{(\lambda x)^i}{i!} \qquad (3.3)$$

From (1.3) we directly obtain that

$$P_A(k) = Er_{k-1}(A) - Er_k(A) = \frac{(\lambda A)^{k-1}}{(k-1)!} e^{-\lambda A}, \qquad (3.4)$$

and from (1.2)

$$D_A(k) = e^{-\lambda A} \left(\frac{(\lambda A)^k}{k!} - \frac{(\lambda A)^{k-1}}{(k-1)!} \right)$$

$$= e^{-\lambda A} \frac{(\lambda A)^{k-1}}{(k-1)!} \left(\frac{\lambda A}{k} - 1 \right) = \lambda e^{-\lambda A} \frac{(\lambda A)^{k-1}}{k!} \left(A - \frac{k}{\lambda} \right), \qquad (3.5)$$

Since, from (3.5) the sign of $D_A(k)$ is positive if and only if $\frac{k}{\lambda} < A$ then $P_A(k)$ attains its maximal value at $k^* + 1$ as claimed. Clearly, $P_A(k)$ is nondecreasing for $k \leq k^* + 1$ and nonincreasing for $k > k^* + 1$. •

3.2 Applications

The next two examples are examined in the context of signal processing based on power accumulation of signals. Basically, signals, whose power is exponentially distributed, are accumulated and checked for threshold detection for a given value A of the threshold. This method is applied in the absence of information about the signals' phase and performance when coherent signal accumulation is impossible. The third example is taken from the field of reliability theory.

Example 1 : Square Law Detector (SLD)

This detector was in use in most radar systems till the Second World War (See Skolnik, 1962). In modern radar systems it is used for sea target detection, for detection of missiles with low-height flight, for resolving the multipass problem and multiple target situation (See e.g. Finn and Johnson, 1968, and Barboy et al. 1986).

Example 2 : Signal Processing in Physiology and Medicine

The same principle of diagnosing detection is used in medical tools intended for researching brain activity (Dawson, 1951).

Example 3 : Reliability (Maintenance and Replacement) (See Barlow and Porshan, 1975.)

Given a system composed of a large number of units each with exponentially distributed life time. Each unit is replaced instantaneously upon failure with an identical unit and the operation of the system continues. In some cases the system has to function properly during a predetermined time period of duration A. In this case we can determine the most likely number of units (Theorem 2) which will be replaced during the system's activity in that period.

4. Normal Distribution

4.1 Mathematical Background

Assume that $r_i; i = 1,2,3,...$ are independent, identical, normally distributed random variables with mean $\mu > 0$ and standard deviation σ.

In this case we prove that the most likely number of these random variables required to cross the given threshold value A is also a function of the parameters μ and σ. We qualify our interpretation of $P_A(k)$ slightly, since (1.3) holds essentially for nonnegative r.v.

First, we prove that the probability $P_A(k)$ as a function of k is a monotonically decreasing function for $k > k^*$.

Theorem 3 : $P_A(k)$ is a monotonically decreasing function of k for any $k > k^* + 1$.

Proof:

From the fact that the sum of independent, normally distributed random variables is itself normally distributed and from (1.3) we obtain the following:

$$P_A(k) = F^{(k-1)}(A) - F^{(k)}(A) =$$

$$= (2\pi)^{-\frac{1}{2}}(k-1)^{-\frac{1}{2}}\sigma^{-1}\int_{-\infty}^{A} e^{-\frac{[x-(k-1)\mu]^2}{2(k-1)\sigma^2}}\,dx$$

$$-(2\pi)^{-\frac{1}{2}}k^{-\frac{1}{2}}\sigma^{-1}\int_{-\infty}^{A} e^{-\frac{(x-k\mu)^2}{2k\sigma^2}}\,dx \qquad (4.1)$$

Then, using the transformations below:

$$A' = \frac{A-(k-1)\mu}{(k-1)^{\frac{1}{2}}\sigma}, \qquad A'' = \frac{A-k\mu}{k^{\frac{1}{2}}\sigma}$$

$$u = \frac{x-(k-1)\mu}{(n-1)^{\frac{1}{2}}\sigma}, \qquad v = \frac{x-k\mu}{k^{\frac{1}{2}}\sigma} \qquad (4.2)$$

$$du = \frac{dx}{(k-1)^{\frac{1}{2}}\sigma}, \qquad dv = \frac{dx}{k^{\frac{1}{2}}\sigma}$$

we obtain

$$P_A(k) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{A'} e^{-\frac{u^2}{2}} du - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{A''} e^{-\frac{v^2}{2}} dv =$$

$$= (2\pi)^{-\frac{1}{2}} \int_{A''}^{A'} e^{-\frac{t^2}{2}} dt \qquad (4.3)$$

Treating k as continuous parameter we differentiate $P_A(k)$ with respect to k to obtain:

$$\frac{dP_A(k)}{dk} = (2\pi)^{-\frac{1}{2}} \left(e^{-\frac{A'^2}{2}} \frac{dA'}{dk} - e^{-\frac{A''^2}{2}} \frac{dA''}{dk} \right), \qquad (4.4)$$

From (4.2)

$$\frac{dA'}{dk} = \frac{-\mu(k-1)^{\frac{1}{2}} - [A-(k-1)\mu]}{(k-1)2(k-1)^{\frac{1}{2}}\sigma} = \frac{-2\mu(k-1) - A + \mu(k-1)}{2(k-1)^{\frac{3}{2}}\sigma} = -\frac{A+\mu(k-1)}{2(k-1)^{\frac{3}{2}}\sigma} \qquad (4.5)$$

and

$$\frac{dA''}{dk} = -\frac{A+\mu k}{2k^{\frac{3}{2}}\sigma}$$

We substitute (4.5) into (4.4) to obtain:

$$\frac{dP_A(k)}{dk} = (2\pi)^{-\frac{1}{2}}(2\sigma)^{-1} \left(e^{\frac{-A'^2}{2}} \frac{A+\mu(k-1)}{(k-1)^{\frac{3}{2}}} + e^{\frac{-A''^2}{2}} \frac{A+\mu k}{k^{\frac{3}{2}}} \right) \qquad (4.6)$$

We assume from now on that $\mu > 0$ and $A > 0$. Since we examine the sequence $\{P_A(k)\}_{k>(k^*+1)}^{\infty}$, i.e., for $k > \frac{A}{\mu} + 1$ we can represent k as:

$$k = \frac{A}{\mu} - m, \quad (4.6a) \qquad \text{where } m + 1 \leq 0.$$

$$A = \mu(k + m), \quad \text{thus}$$

$$A' = \frac{\mu(k+m) - \mu(k-1)}{\sigma(k-1)^{\frac{1}{2}}} = \frac{\mu(m+1)}{\sigma(k-1)^{\frac{1}{2}}}$$

$$A'' = \frac{\mu m}{\sigma k^{\frac{1}{2}}} \qquad (4.7)$$

The equation (4.6) is rewritten as follows:

$$\frac{dP_A(k)}{dk} = (2\pi)^{-\frac{1}{2}}(2\sigma)^{-1} \left(e^{-\frac{\mu^2 m^2}{2\sigma^2 k}} \mu(2k + m)k^{-\frac{3}{2}} \right.$$

$$\left. - e^{-\frac{\mu^2(m+1)^2}{2\sigma^2(k-1)}} \mu(2k + m - 1)(k - 1)^{-\frac{3}{2}} \right)$$

$\frac{dP_A(k)}{dk} < 0$ if and only if the following inequality holds:

$$e^{-\frac{\mu^2(m+1)^2}{2\sigma^2(k-1)}}(2k+m-1)(k-1)^{-\frac{3}{2}} > e^{-\frac{\mu^2 m^2}{2\sigma^2 k}}(2k+m)k^{-\frac{3}{2}}$$

or equivalently

$$e^{-\frac{\mu^2}{2\sigma^2}\left(\frac{(m+1)^2}{k-1}-\frac{m^2}{k}\right)} > \frac{2k+m}{2k+m-1}\left(\frac{k-1}{k}\right)^{\frac{3}{2}}, \qquad (4.8)$$

The proof for inequality (4.8) consists of two parts. First, we prove that the left hand side of (4.8) is greater than 1 and then prove that the right hand side of (4.8) is less than 1.

In order to prove the first part it is sufficient to show that $\frac{(m+1)^2}{k-1} - \frac{m^2}{k} < 0$

or equivalently;

$$m^2 + 2km + k < 0, \qquad (4.9)$$

or $(k+m)^2 < k(k-1)$

Since $-k < m$ and $m+1 \leq 0$, therefore $k+m \leq k-1$; thus, (4.9) holds and the left hand side of (4.8) is greater than 1.

Next, we show that the right hand side of (4.8) is less than 1.

$$\frac{2k+m}{2k+m-1}\left(\frac{k-1}{k}\right)^{\frac{3}{2}} = \left(1+\frac{1}{2k+m-1}\right)\left(\frac{k-1}{k}\right)^{\frac{3}{2}}$$

The above expression obtains its maximal value when m is minimal,

i.e., for $m = -k$. Therefore,

R.H.S.$\leq \left(1+\frac{1}{2k-k-1}\right)\left(\frac{k-1}{k}\right)^{\frac{3}{2}} = \left(\frac{k-1}{k}\right)^{\frac{1}{2}}$ which is always less than 1.

This concludes the our proof of Theorem 3 •

Corollary : The most likely number of independent, identical, normally distributed random variables to cross a given threshold value A falls somewhere between 1 and $k^* + 1$.

A more specific result is obtained when restricting the coefficient of variation of the underlying normal distribution to less than or equal to one. I.e., $c = \frac{\sigma}{\mu} \leq 1$.

In Dror, 1983 and in Dror and Ball, 1987 it is proven that when restricting $c \leq 1$ the sequence of $\{P_A(k)\}_{k=1}^{k}$ is a monotonically increasing function in k. This result together with the result proven here in Theorem 3 combine to prove the following theorem:

Theorem 4 : Given that the random variables r_i are identical, independent, normally distributed random variables with a coefficient of variation less than or equal to one, then $max_k P_A(k)$ is obtained at k^* or $k^* + 1$.

In order to show that the result from Theorem 4 does not hold in the general case, we prove the following theorem:

Theorem 5 : Given that the coefficient of variation $c \gg 1$ for the random variables in Theorem 4, then the sequence $\{P_A(K_k)\}_{k \leq k^*}$ is monotonically decreasing in k. When $lim_{c \to \infty} \to \infty$ then $max_{k \leq k^*} P_A(k) = P_A(1)$.

Proof : Examine the expression (4.8) for nonnegative values of m.

First, we show that the R.H.S. of (4.8) is strictly less than 1.

$$\text{R.H.S.} = \left(1 + \frac{1}{2k+m-1}\right)\left(\frac{k-1}{k}\right)^{\frac{3}{2}} \leq \left(1 + \frac{1}{2k-1}\right)\left(\frac{k-1}{k}\right)^{\frac{3}{2}} =$$

$$= \frac{2k}{2k-1}\frac{k-1}{k}\left(\frac{k-1}{k}\right)^{\frac{1}{2}} = \frac{2k-2}{2k-1}\left(\frac{k-1}{k}\right)^{\frac{1}{2}} < 1$$

On the other hand, the L.H.S. of (4.8) though also less than 1, can approach 1 as close as desired by making the coefficient of variation c as big as necessary.

In order to show that the L.H.S. of (4.8) is less than 1 it is sufficient to show that for $m \geq 0$ the expression $\frac{m+1^2}{k-1} - \frac{m^2}{k} > 0$; for nonnegative values of m and k this is obvious.

Now, in order to observe that the L.H.S. of (4.8) can approach 1 from below as close as desired it is sufficient to see that as $c \to \infty$ this expression approaches $\frac{1}{e^0}$. Thus, since the R.H.S. of (4.8) is strictly less than 1 and the L.H.S. can be made very close to 1, the inequality (4.8) is preserved. Since it can be made true for any k; thus, by appropriately large c it holds true for $k = 1$•

4.2 Applications

Example 1:

The actual application of the result derived in Theorem 4 is fully described in Dror, 1983 and in Dror and Ball, 1987. It proved to be a very helpful result when designing detailed operational activities for what is called 'The Inventory Routing Problem'-(IRP). We present here a very abbreviated summary of the successful application of the monotonicity characteristics for $P_A(k)$.

In the above IRP a set of customers (approximately 3000) consume daily (continuously) a certain commodity (heating oil). The customers daily consumption rates are assumed to normally distributed. The seasonality effects are incorporated in the problem's solution as well. A customer's supply of heating oil is stored in tanks of fixed capacity. (Different customers might have different tank capacities.) These storage tanks are refilled by a fleet of special vehicles operating from one central depot with unlimited supply. The company operating the vehicles (i.e.,the supplier of the heating oil) has a contructual agreement with the customers to keep them

always supplied with this basic commodity. The company's objective is to perform this distribution operation efficiently. There are two main cost factors influencing the efficiency of this distribution operation: One represents the cost of servicing a customer when his tank still contains some amount of the commodity, and the other represents the cost when a customer stuck-out and a special delivery is required in order to fill the customer's tank. This special delivery cost is much higher than the cost of a regular service. On one hand, the company does not want to service a customer very often since the regular service costs add up. On the other hand, scheduling a service too late increases the likelihood of incurring the cost of special delivery. The sequence $\{P_A(k)\}_k$ expresses on this case the probability of stock-out on exactly day k and the necessity of special delivery to a customer i with consumption rate r_i and a storage tank of capacity A. Establishing the monotonicity characteristic of this sequence $\{P_A(k)\}_k$ enables the analyst to construct a combined convex cost function which incorporates both types of costs effectively. The minimum value of such cost function determines the desired customer's replenishment date and at the same time the customer's expected safety stock level.

Example 2:

In Dror and Trudeau, 1986, a stochastic vehicle routing problem is described. A dispatcher has to design a number of vehicle routes in a situation where the actual customer demand is known only upon the vehicle's arrival at the customer location. Again, the monotonicity characteristics of the $P_A(k)$ are observed in the case of customers demands being normally distributed, and this property is incorporated in a very efficient heuristic routing procedure.

5. Poisson Distribution

5.1 Mathematical Background

Assume that r_i are i.i.d.rv. distributed according to Poisson density with parameter $\lambda > 0$, i.e.,

$$Prob.(r_i = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad t \geq 0 \tag{5.1}$$

Its mean, variance and coefficient of variation are given respectively by

$$\mu = \lambda t,\ \sigma^2 = \lambda^2 t^2, \text{ and } c = \frac{\sigma}{\mu} = 1 \tag{5.2}$$

Since r_i is integer-valued, we will consider integer A only.

Theorem 6 : Given that the i.i.d.r.v. r_i are distributed according to Poisson pdf (5.1), then the probability $P_A(k)$ defined by (1.1) attains its maximum either at $k = k^*$ or $k = k^* + 1$ where k^* is given by $k^*\lambda t \leq A < (k^* + 1)\lambda t$. Moreover, $P_A(k)$ is nondecreasing for $k < k^*$ and nonincreasing for $k > k^* + 1$.

Proof : We will use the fact that the sum od several independent Poisson distributed random variables is also Poisson distributed and its parameter is equal to sum of components' parameters. Thus,

$$F^{(k-1)}(A) = Prob.\{\sum_{i=1}^{k-1} r_i \leq A\} = \sum_{n=0}^{A} e-\lambda(k-1)t \frac{[\lambda(k-1)t]^n}{n!}, \qquad (5.3)$$

$$F^{(k)}(A) = Prob.\{\sum_{i=1}^{k} r_i \leq A\} = \sum_{n=0}^{A} e^{-\lambda kt} \frac{(\lambda kt)^n}{n!}, \qquad (5.4)$$

and $P_A(k) = F^{(k-1)}(A) - F^{(k)}(A) = \sum_{n=0}^{A}\{e^{-\lambda(k-1)t} \frac{[\lambda(k-1)t]^n}{n!} - e^{-\lambda kt} \frac{[\lambda kt]^n}{n!}\}$ (5.5)

Note that for Poisson process N_t (see e.g. Cinlar, 1975),

$$Prob.\{N_t \leq A\} = Prob.\{T_{A+1} \geq t\} = \sum_{n=0}^{A} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \qquad (5.6)$$

where T_{A+1} has an Erlang - (A+1) distribution with parameter γ.

Differentiating $Prob.\{T_{A+1} \leq t\} = 1 - \sum_{n=0}^{A} e^{-\gamma t} \frac{(\gamma t)^n}{n!}$ (5.7) with respect to t we obtain the corresponding pdf

$$\frac{\gamma(\gamma t)^A e^{-\lambda t}}{A!} \qquad (5.8)$$

It is interesting to note that k and t enter the formulas (5.3) and (5.4) in a similar way, therefore differentiating with respect to k is similar to differentiating with respect to t. Keeping this fact in mind together with (5.7)-(5.8), we differentiate (5.5) with respect to k, namely

$$\frac{dP_A(k)}{dk} = \frac{\lambda t(\lambda tk)^A e^{-\lambda tk}}{A!} - \frac{\lambda t[\lambda t(k-1)]^A e^{-\lambda t(k-1)}}{A!} =$$

$$= \frac{(\lambda t)^{A+1} e_\lambda tk}{A!}[k^A - (k-1)^A e^{\lambda t}] \qquad (5.9)$$

$\frac{dP_A(k)}{dk} = 0$ if and only if $k^A - (k-1)^A e^{\lambda t} = 0$

or equivalently, $(\frac{k-1}{k} = \epsilon^{-\lambda t}$

Therefore, $\frac{1}{k} = 1 - e^{-\frac{\lambda t}{A}}$ (5.10)

Using Taylor's series expansion of exponent and noting that $\frac{\lambda t}{A} < 1$, we obtain

$$\frac{1}{k} = \frac{\lambda t}{A} + O(\frac{\lambda t}{A}) \qquad (5.11)$$

Thus, $k \sim \frac{A}{\lambda t}$

Therefore, since k is an integer, $P_(k)$ attains its maximum either at $k = k^*$ or $k = k^* + 1$.

Obviously, if $k < k^*$, then $\frac{dP_A(k)}{dk} > 0$, i.e., $P_A(k)$ increases, and if $k > k^* + 1$,

then $\frac{dP_A(k)}{dk} < 0$, i.e., $P_A(k)$ decreases •

5.2 Applications

Example 1 : Inventory

Consider the inventory system (warehouse) which operates according to (s, S) policy with demand r_i distributed according to Poisson density with parameter λ units/day. Given that the warehouse orders a new shipment of $A = S - s$ units when its inventory is less than the threshold level s, $k = k^*$ (or $k = k^* + 1$) is the most likely number of days between successive renewal of inventory.

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